# Hamel flow of certain anisotropic fluids

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An exact solution is given for the flow in a convergent or divergent channel of a class of anisotropic fluids in which the fluid has a preferred direction.

## 1. Introduction

Ericksen (1960*a*) has formulated a properly invariant theory of anisotropic fluids in which he considered an incompressible fluid having at each particle a single preferred direction. This direction, which is governed by the fluid motion, can vary throughout the fluid and with time. The fluid may be thought of as a suspension of particles which are bodies of revolution, the axis of revolution giving the preferred direction.

Ericksen assumed that the stress at a point is a function of the velocity gradients and the preferred direction at that point. He used well-known invariance principles to show that the stress tensor of the components of stress $t_{ij}$ (Cartesian tensor notation is used) is related to the velocity gradients and the preferred direction as follows:

$$t_{ij} = -p\delta_{ij} + \alpha_1 n_i n_j + \alpha_2 d_{ij} + \alpha_3 d_{ik} d_{kj} + \alpha_4(n_i n_k d_{kj} + n_j n_k d_{ki}) + \alpha_5(n_i n_k d_{km} d_{mj} + n_j n_k d_{km} d_{mi}). \quad (1)$$

where $p$ is a scalar function of the space variables and time, $\delta_{ij}$ the Kronecker delta, $n_i$ the unit vector giving the preferred direction and $d_{ij}$ the rate of strain tensor. If the velocity vector is $v_i$, $2d_{ij} = v_{i,j} + v_{j,i}$. The $\alpha$'s are functions of the invariants

$$n_i n_i, \quad d_{ij} n_i n_j, \quad d_{ik} d_{kj} n_i n_j, \quad d_{ij} d_{ij}, \quad d_{ik} d_{kj} d_{ji}. \quad (2)$$

Ericksen also assumed that the material derivative of the preferred direction is a function of the velocity gradients and the preferred direction. He used standard invariance procedures to obtain the equation

$$\dot{n}_i = \omega_{ij} n_j + \beta_2(d_{ij} n_j - d_{jk} n_j n_k n_i) + \beta_3(d_{ik} d_{kj} n_j - d_{km} d_{mj} n_k n_j n_i), \quad (3)$$

where $\dot{n}_i$ is the material derivative of the unit vector $n_i$ and $2\omega_{ij} = v_{i,j} - v_{j,i}$. The $\beta$'s are functions of the invariants (2).

Ericksen simplified the above equations by considering the case when they are linear in the rate-of-strain tensor $d_{ij}$. Equations (1) and (3) reduce to

$$t_{ij} = -p\delta_{ij} + 2\mu d_{ij} + (\mu_1 + \mu_2 d_{km} n_k n_m) n_i n_j + 2\mu_3(d_{jk} n_k n_i + d_{ik} n_k n_j), \quad (4)$$

and

$$\dot{n}_i = \omega_{ij} n_j + \lambda(d_{ij} n_j - d_{kj} n_k n_j n_i), \quad (5)$$

where $\lambda$ and the $\mu$'s are constants. In the absence of body forces, the equations of motion are

$$t_{ij,j} = \rho \dot{v}_i, \tag{6}$$

where $\rho$ is the density of the fluid. Since the fluid is incompressible, the continuity equation takes the form

$$d_{ii} = 0. \tag{7}$$

Subsequently Ericksen (1960*b*, 1960*c*, 1962) gave a more general theory and considered various aspects of it. Hand (1962) and Green (1964) have also given theories of anisotropic fluids.

Exact solutions of the equations (4)–(7) have been given by Ericksen (1960*a*, 1961) for simple shear flow and Poiseuille flow and by Verma (1962) for Couette flow. In order to obtain steady solutions of those equations only values of the parameter $\lambda$ such that $|\lambda| > 1$ were considered.

If the fluid is at rest (or is undergoing rigid-body motions), the components of the tensor $d_{ij}$ are zero and the stress components are given by

$$t_{ij} = -p\delta_{ij} + \mu_1 n_i n_j.$$

In general this is not a hydrostatic pressure and the fluid at rest sustains a shear stress. Thus constitutive equations with non-zero $\mu_1$ describe materials of the Bingham type (cf. Ericksen 1961). Here constitutive equations with $\mu_1$ zero are considered.

An exact solution which has aided the understanding of the behaviour of Newtonian fluids is that by Hamel (see Goldstein 1938) for flow in a convergent or divergent channel. A similar exact solution is given below for anisotropic fluids with constitutive equations (4) and (5) (with $\mu_1 = 0$). The problem reduces to the solution of two ordinary differential equations, and, as an example, these are integrated numerically in a special case. It is found that there are two solutions which satisfy the necessary boundary conditions. Following Ericksen's arguments with regard to stability in which perturbations of the vector $n_i$ are considered while the velocity remains unperturbed, one of these solutions appears to be stable and the other unstable. However, this question is not investigated in greater detail here.

Solutions of equations (4)–(7) are readily found for the flow between plane parallel walls. The solution which appears to be stable has a discontinuity in the preferred direction across the plane of zero shear midway between the walls. It is thought of interest to compare this solution with that for flow in a channel of very small angular gap. Accordingly the equations are again integrated in a special case when the walls are almost parallel. In the stable solution obtained there is rather an abrupt change in the preferred direction in the centre of the channel in good agreement with the solution for flow between parallel walls.

## 2. The solution for flow in a convergent or divergent channel

All tensor quantities are expressed in terms of their physical components in cylindrical polar co-ordinates $(r, \theta, z)$ which are chosen so that the walls of the channel coincide with the planes $\theta = \pm\theta_0$, where $\theta_0$ is a constant.

Solutions are examined in which the velocity is of the form

$$v_r = r^{-1}f(\theta), \quad v_\theta = v_z = 0,$$

and in which the preferred direction is given by

$$n_r = \cos\phi, \quad n_\theta = \sin\phi, \quad n_z = 0,$$

where $\phi$ is a function of $\theta$. The non-zero components of the rate of strain and rotation tensors are

$$d_{rr} = -f/r^2, \quad d_{\theta\theta} = f/r^2,$$

$$d_{r\theta} = d_{\theta r} = f'/2r^2, \quad \omega_{r\theta} = -\omega_{\theta r} = f'/2r^2,$$

where an accent denotes differentiation with respect to $\theta$. Equation (4) gives the stress components

$$\begin{aligned} t_{rr} &= -p+(1/r^2)[-2\mu f+\mu_2\Theta\cos^2\phi+\mu_3(f'\sin 2\phi-4f\cos^2\phi)], \\ t_{\theta\theta} &= -p+(1/r^2)[2\mu f+\mu_2\Theta\sin^2\phi+\mu_3(f'\sin 2\phi+4f\sin^2\phi)], \\ t_{r\theta} &= (1/r^2)[(\mu+\mu_3)f'+\mu_2\Theta\sin\phi\cos\phi], \\ t_{zz} &= -p, \quad t_{rz} = t_{\theta z} = 0, \end{aligned}$$

where

$$\Theta = \tfrac{1}{2}f'\sin 2\phi - f\cos 2\phi.$$

The angle $\phi$ is found in terms of the fluid motion from equation (5) which reduces to

$$f'(1-\lambda\cos 2\phi) = 2\lambda f\sin 2\phi. \tag{8}$$

At the walls where $f$ is zero on account of the no-slip condition

$$\cos 2\phi = 1/\lambda. \tag{9}$$

As in earlier work only values of $\lambda$ such that $|\lambda| > 1$ are considered.

Since $v_r$ is the only non-vanishing component of velocity and the solution is independent of $z$, the equations of motion become

$$\rho v_r \frac{\partial v_r}{\partial r} = \frac{\partial t_{rr}}{\partial r} + \frac{1}{r}\frac{\partial t_{r\theta}}{\partial\theta} + \frac{t_{rr}-t_{\theta\theta}}{r}, \tag{10}$$

and

$$0 = \frac{\partial t_{r\theta}}{\partial r} + \frac{2t_{r\theta}}{r} + \frac{1}{r}\frac{\partial t_{\theta\theta}}{\partial\theta}. \tag{11}$$

From equation (11) the pressure is given by

$$p = k(r) + (1/r^2)[2\mu f+\mu_2\Theta\sin^2\phi+\mu_3(f'\sin 2\phi+4f\sin^2\phi)],$$

where $k(r)$ is an arbitrary function of the variable $r$. Equation (10) reduces to

$$\begin{aligned} &(\mu+\mu_3+\tfrac{1}{4}\mu_2\sin^2 2\phi)f'' - \tfrac{1}{2}(\mu_2\sin 4\phi)f' + \tfrac{1}{2}(\mu_2\sin 4\phi)f'\phi' \\ &\qquad -(\mu_2\cos 4\phi)f\phi' + 4(\mu+\mu_3+\tfrac{1}{4}\mu_2\cos^2 2\phi)f + \rho f^2 = \beta, \end{aligned} \tag{12}$$

and

$$k(r) = p_0 - \beta/2r^2,$$

where $p_0$ and $\beta$ are constants. The boundary conditions are

$$\left.\begin{aligned} &f(\theta_0) = 0, \quad f(-\theta_0) = 0, \\ \text{and} \quad &2\phi(\theta_0) = \cos^{-1}(1/\lambda), \quad 2\phi(-\theta_0) = \cos^{-1}(1/\lambda). \end{aligned}\right\} \tag{13}$$

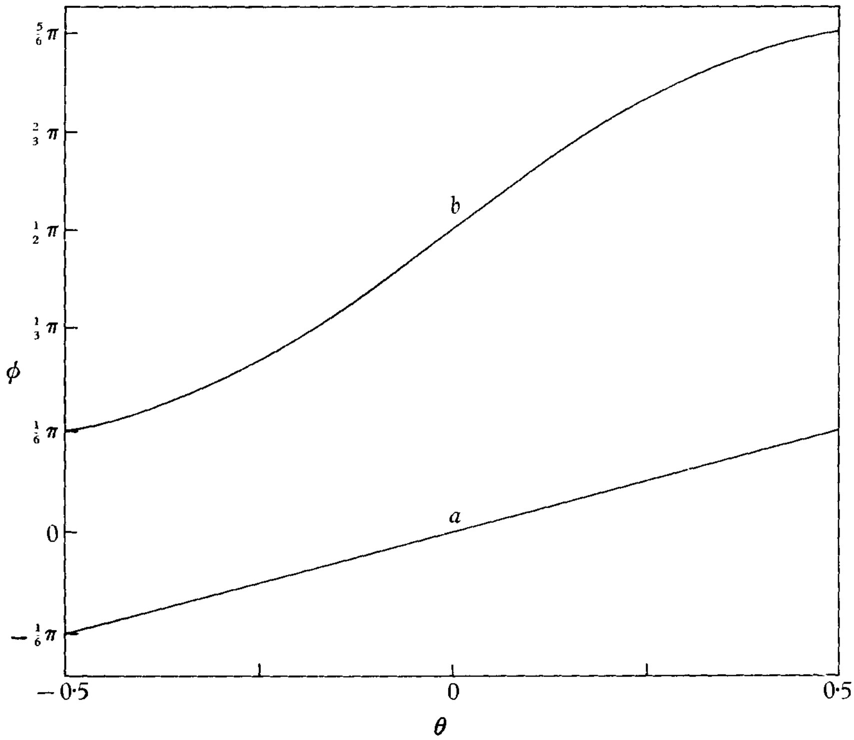


FIGURE 1. The angle $\phi$ as a function of $\theta$ for the two solutions, $a$ and $b$, when $\theta_0 = 0{\cdot}5$ rad and $\mu + \mu_3 = 2\mu_2$, $\lambda = 2$, $\beta = 2{\cdot}5(\mu+\mu_3)^2/\rho$.

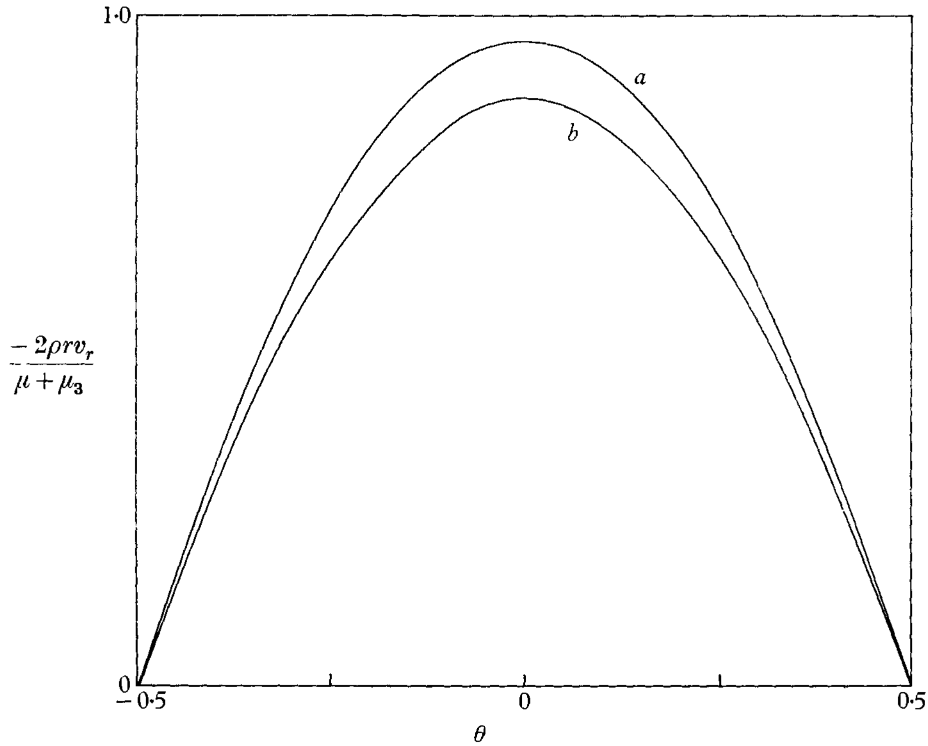


FIGURE 2. The velocity profiles $-2\rho r v_r/(\mu+\mu_3)$ for the two solutions, $a$ and $b$, when $\theta_0 = 0{\cdot}5$ rad and $\mu+\mu_3 = 2\mu_2$, $\lambda = 2$, $\beta = 2{\cdot}5(\mu+\mu_3)^2/\rho$.

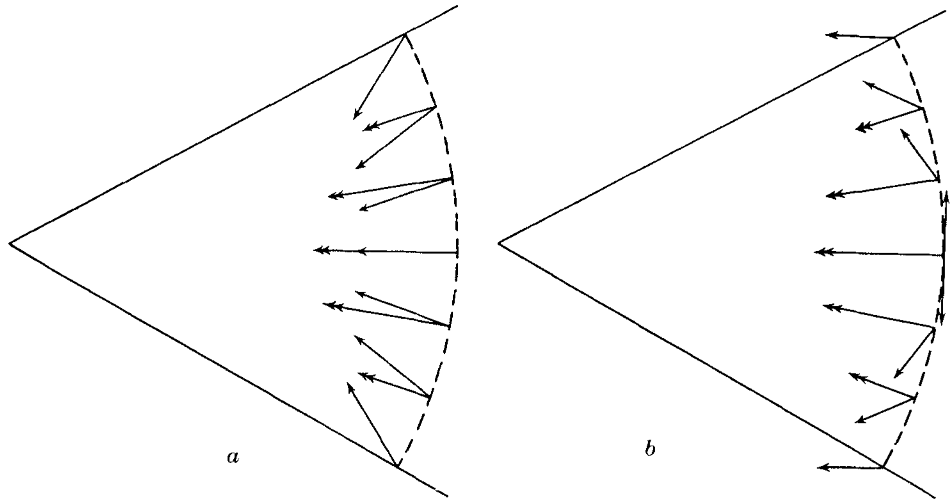


FIGURE 3. ($a$) Solution $a$ ($a$ and $b$ correspond to the relevant $\phi$ solutions in figure 1): ←←— velocity vector, ←—— preferred direction. ($b$) Solution $b$: ←←— velocity vector, ←—— preferred direction.

As an example equations (8) and (12) are integrated on a computer subject to the boundary conditions (13) when

$$\mu+\mu_3 = 2\mu_2, \quad \lambda = 2,$$
$$\beta = 2{\cdot}5(\mu+\mu_3)^2/\rho \quad \text{and} \quad \theta_0 = 0{\cdot}5\,\text{rad}.$$

The results are shown in figures 1–3. Two solutions are found corresponding to the two values of $\phi$ given by equation (9). It should be noted that the theory does not distinguish between the vectors $n_i$ and $-n_i$ for the preferred direction, i.e. between the angles $\phi$ and $\phi \pm \pi$. In figure 1 the functions $\phi$ are given for solution $a$ in which $\phi$ is $\frac{1}{6}\pi$ at $\theta = 0{\cdot}5$ rad, and for solution $b$ in which $\phi$ is $\frac{5}{6}\pi$ at $\theta = 0{\cdot}5$ rad. In figure 2 the velocity profiles are given. The solutions are illustrated diagrammatically in figure 3 by drawing the velocity and preferred direction vectors at a series of points at a fixed radial distance. In solution $a$ the angle between the preferred direction and the velocity vector decreases from a maximum at the walls to zero at the centre of the channel. In solution $b$ the angle between the preferred direction and the velocity vector increases from a minimum at the wall to $\frac{1}{2}\pi$ at the centre. Both solutions are symmetrical about the plane $\theta = 0$.

To examine the sensitivity of those solutions to the parameter $\mu_2/(\mu+\mu_3)$ a second integration is performed when $\lambda$, $\beta$ and $\theta_0$ are unchanged but $2\mu_2 = 5(\mu+\mu_3)$. Two solutions are again found similar to those above. The corresponding values of $\phi$ show little change (of order 1 %) while the velocity profiles are reduced by a factor of 10 % in solution $a$ and 30 % in solution $b$. Comparison with the corresponding solution for a Newtonian fluid of viscosity $\mu$ shows that the velocity profiles are similar to the Newtonian profile but smaller by a factor of about one-half.

When analysing simple shear flow, Ericksen found more than one solution but discarded all but one on the grounds of a simple stability argument in which he allowed perturbations in the vector $n_i$ while the velocity vector remained unperturbed. Following this procedure for small perturbations the solution $a$ is found to be stable and solution $b$ unstable.

## 3. Comparison with flow between parallel walls

Solutions are obtained for equations (4)–(7) for the flow between parallel walls. Cartesian co-ordinates ($x$, $y$, $z$) are chosen such that the walls coincide with the planes $y = \pm h$ where $h$ is an arbitrary constant. Solutions are examined in which the velocity has the form

$$v_x = g(y), \quad v_y = v_z = 0,$$

and the preferred direction is given by

$$n_x = \cos\phi, \quad n_y = \sin\phi, \quad n_z = 0,$$

where $\phi$ may vary with $y$. From equation (4)

$$\begin{aligned} t_{xx} &= -p + g' \sin\phi\cos\phi(2\mu_3 + \mu_2\cos^2\phi), \\ t_{yy} &= -p + g' \sin\phi\cos\phi(2\mu_3 + \mu_2\sin^2\phi), \\ t_{xy} &= g'(\mu + \mu_3 + \mu_2\sin^2\phi\cos^2\phi), \\ t_{zz} &= -p, \quad t_{xz} = t_{yz} = 0, \end{aligned}$$

where an accent denotes differentiation with respect to $y$. Equation (5) reduces to

$$g' \sin\phi(1-\lambda\cos 2\phi) = 0, \quad g' \cos\phi(1-\lambda\cos 2\phi) = 0.$$

If $g' \neq 0$, $\cos 2\phi = 1/\lambda$, giving two possible values of $\phi$. When $\lambda > 1$, the stable solution appears to be

$$\begin{aligned} \phi &= \tfrac{1}{2}\cos^{-1}(1/\lambda), \quad 0 < \phi < \tfrac{1}{4}\pi \quad \text{if} \quad g' > 0, \\ \phi &= \tfrac{1}{2}\cos^{-1}(1/\lambda), \quad \tfrac{3}{4}\pi < \phi < \pi \quad \text{if} \quad g' < 0, \end{aligned}$$

and when $\lambda < -1$,

$$\begin{aligned} \phi &= \tfrac{1}{2}\cos^{-1}(1/\lambda), \quad \tfrac{1}{4}\pi < \phi < \tfrac{1}{2}\pi \quad \text{if} \quad g' < 0, \\ \phi &= \tfrac{1}{2}\cos^{-1}(1/\lambda), \quad \tfrac{1}{2}\pi < \phi < \tfrac{3}{4}\pi \quad \text{if} \quad g' > 0. \end{aligned}$$

The equations of motion and the boundary conditions are satisfied by

$$p = p_0 - Px + g' \sin\phi\cos\phi(2\mu_3 + \mu_2\sin^2\phi),$$

and

$$g(y) = P(h^2 - y^2)/2(\mu + \mu_3 + \mu_2\sin^2\phi\cos^2\phi),$$

where $P$ and $p_0$ are constants. At $y = 0$, $g'$ changes sign, and thus there is a discontinuity in $\phi$ at that point. However, $v_x$, $t_{yx}$, $t_{yy}$ and $t_{yz}$ are all continuous across $y = 0$.

As $\theta_0$ tends to zero one feels intuitively that the solution for flow in the convergent or divergent channel must approach the solution for flow between parallel walls. Indeed for a Newtonian fluid this can be shown to be the case analytically. Consequently a second integration of equations (8) and (12) with boundary conditions (13) is performed in the special case

$$\mu + \mu_3 = 2\mu_2, \quad \lambda = 2,$$

$$\beta = 25(\mu + \mu_3)^2/\rho, \quad \text{and} \quad \theta_0 = 0{\cdot}05\,\text{rad}.$$

Two solutions are found similar to those obtained in the previous section. The values of $\phi$ are shown in figure 4 and it can be seen that there is a rapid change in value near the centre of the channel. The velocity profiles are found to be very closely parabolic. Again the solution where $\phi$ is $\frac{1}{6}\pi$ at $\theta = 0{\cdot}05$ rad appears to be the stable solution and this is in good agreement with the solution given above for flow between parallel walls.

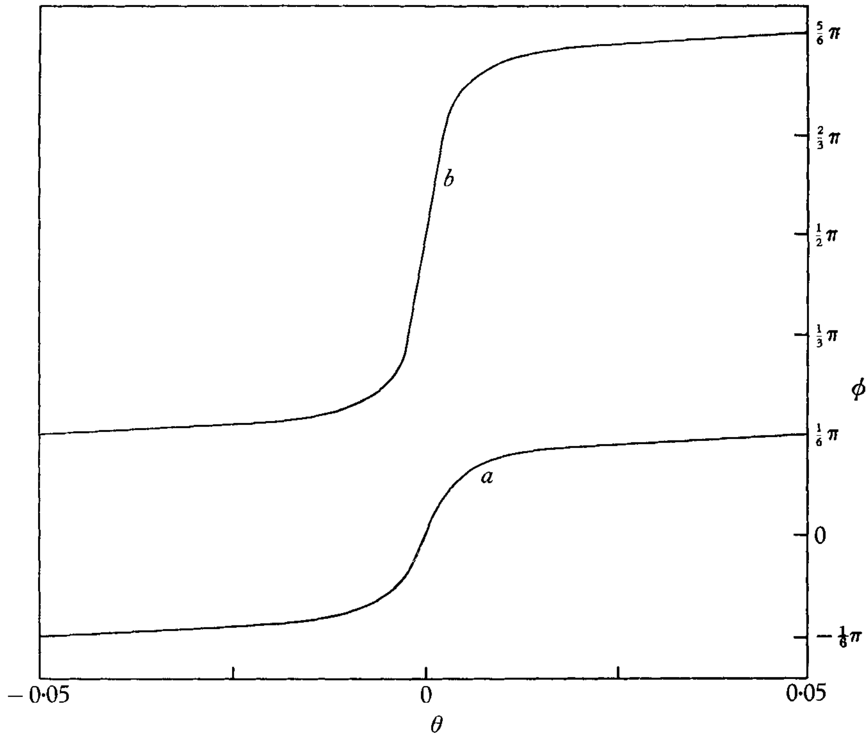


FIGURE 4. The angle $\phi$ as a function of $\theta$ for the two solutions, $a$ and $b$, when $\theta_0 = 0{\cdot}05$ rad and $\mu+\mu_3 = 2\mu_2$, $\lambda = 2$, $\beta = 25(\mu+\mu_3)^2/\rho$.

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